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On the Lie symmetries of the classical Kepler problem

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Abstract. The standard version of Noether's theorem, when applied to the classical Kepler problem, leads to the constants of energy and angular momentum, but does not give the '*hidden symmetry*' known as the Runge–Lenz vector. Lie's theory of differential equations is used to obtain all three constants of motion. The transformations of solutions under the point transformations to which these constants correspond are studied. The results are generalised to n dimensions.

1. Introduction

Conservation laws play an important role in providing an adequate mathematical description of dynamical systems and in uncovering their symmetries.

In a recent paper the authors obtained the Lie and Noether symmetry groups of the time-dependent oscillator in n dimensions. In this work we examine the analogous groups for the classical Kepler problem.

The classical Kepler problem has merited much investigation because of its role in quantum mechanics as well as its inherent importance in classical mechanics. Three constants of motion arise: the energy, the angular momentum and the Runge–Lenz vector. This last quantity is a vector that is parallel to the line joining the centre of force to the nearer apse, and is a measure of the eccentricity of the orbit. The symmetry associated with this vector has been called a 'hidden symmetry' of the problem, in that its existence is not immediately apparent from an inspection of the geometric symmetries of the force field (Cisneros and McIntosh 1970).

When a dynamical system has a Lagrangian formulation, Noether's theorem is commonly used to find constants of motion. This theorem enables a constant of motion to be constructed for each element of a group of point transformations leaving the action integral invariant. There are however some limitations: the Runge–Lenz vector, for example, evades detection by Noether's theorem. The problem is partly alleviated by noting that the Lagrangian for a given problem may vary up to a total time derivative without altering the variational problem and thus the equation of motion. Even with this generalised Noether's theorem some of the 'hidden symmetries' in classical problems still do not appear. To take a case in point, the Runge–Lenz vector will only arise from a Noether-type theorem when the transformations applied to the action integral are not point transformations but involve the velocity as an independent variable (Hill 1951, Lévy-Leblond 1971). With the introduction of these types of transformations, some of the features of the theory are no longer straightforward (Gonzalez-Gascon 1977).

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In an earlier paper (Prince and Eliezer 1980) the authors pointed out that another 'hidden symmetry' in classical mechanics, the Fradkin–Hill matrix $\dot{x}_k\dot{x}_l + x_k x_l$ for the n-dimensional simple harmonic oscillator, could be obtained from Noether's theorem in its original form. It has previously been obtained using velocity-dependent transformations (Lévy-Leblond 1971). It would be satisfying to obtain the Runge–Lenz vector of the Kepler problem by some method involving point transformations only.

Recently there has been a revival of interest in Lie's theory of differential equations, particularly as applied to symmetry considerations of equations of motions of dynamical systems (Wulfman and Wybourne 1976, Lutzky 1978, Prince and Eliezer 1980). This theory in its standard form (Lie 1891, 1922, Page 1897, Cohen 1911, Ince 1926, Bluman and Cole 1974) considers the invariance of solutions of a differential equation (and thus invariance of the form of the differential equation itself) under point transformations of one parameter. Lie himself showed that for a second-order equation of the form

$$y'' + f(x, y, y') = 0 \tag{1}$$

there are at most eight such point transformations. He further gave the form of these for the free particle. The simple harmonic oscillator (Anderson and Davison 1974, Wulfman and Wybourne 1976, Lutzky 1978) and the time-dependent oscillator (Prince and Eliezer 1980) also admit eight transformations in this way. For the oscillator problems, the five one-parameter transformations associated with the standard Noether problem form a subgroup of the eight for the Lie method.

An approach was outlined in Prince (1979) by which constant of motion can be calculated for each transformation admitted by the equation of motion in a given problem. Eight such constants were constructed for the oscillator problem, only five of which were previously available from Noether's theorem. In the present work we show that a point transformation in the Lie group of the Kepler problem provides the Runge–Lenz vector.

Firstly we outline the features of the Kepler problem and then the Lie and Noether groups are determined. The constants are calculated in § 4. The results are then generalised to n dimensions.

2. The Kepler problem in classical mechanics

The equation governing the motion of a particle in an inverse square force field in E_3 is

$$\ddot{\boldsymbol{r}} + \mu \boldsymbol{r}/r^3 = 0 \tag{2}$$

where μ is a constant and $r = |\boldsymbol{r}|$. Equation (2) may be obtained from an action integral formulation with Lagrangian

$$\mathscr{L} = \tfrac{1}{2}\dot{\boldsymbol{r}}^2 + \mu/r. \tag{3}$$

Constants of motion are

$$E = \tfrac{1}{2}\dot{\boldsymbol{r}}^2 - \mu/r, \tag{4}$$

$$\boldsymbol{L} = \boldsymbol{r} \times \dot{\boldsymbol{r}}, \qquad \boldsymbol{R} = \dot{\boldsymbol{r}} \times \boldsymbol{L} - \mu\boldsymbol{r}/r. \tag{5}$$

E is the total energy, $\boldsymbol{L}$ is the angular momentum and $\boldsymbol{R}$ is known as the Runge–Lenz vector. Collinson (1973) develops the details of the motion from (3), (4), (5) rather succinctly, as follows. $\boldsymbol{L} \cdot \boldsymbol{r} = 0$ and so the particle moves in a plane perpendicular to $\boldsymbol{L}$. Since $\boldsymbol{R} \cdot \boldsymbol{L} = 0$ the Runge–Lenz vector lies in the plane of motion. Calling $|\boldsymbol{L}| = h$,

$$\boldsymbol{r} \cdot \boldsymbol{R} = h^2 - \mu r. \tag{6}$$

If θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{R}$ then (6) becomes

$$r = \frac{h^2/\mu^2}{1 + (R/\mu)\cos\theta}, \tag{7}$$

the equation of a conic with the axis in the direction $\theta = 0$. The semi-latus rectum l and eccentricity ε of the conic are given by

$$l = h^2/\mu, \tag{8}$$

$$\varepsilon = R/\mu. \tag{9}$$

Collinson also mentions that Hamilton knew about this sort of thing in 1845 well before the derivations of Runge and Lenz this century. Briefly, Hamilton had obtained a vector

$$\boldsymbol{W} = \dot{\boldsymbol{r}} - \mu(\hat{\boldsymbol{r}} \times \boldsymbol{L})/h^2. \tag{10}$$

The Runge–Lenz vector is $\boldsymbol{W} \times \boldsymbol{L}$. The usual equation of the planetary orbit can be obtained by taking the vector product with $\boldsymbol{r}$ and choosing θ to be $\pi/2$ minus the angle between $\boldsymbol{W}$ and $\boldsymbol{r}$.

We see that the geometric and dynamical characteristics of the motion are related as follows:

plane of the orbit —— direction of angular momentum;
orientation of the orbit —— direction of the Runge–Lenz vector in the plane;
semi-latus rectum —— magnitude of angular momentum;
eccentricity —— magnitude of the Runge–Lenz vector.

With this interpretation the energy does not appear as an independent quantity but is given by

$$E = (R^2 - \mu^2)/2h^2. \tag{11}$$

The particular type of conic in a given problem may be determined as $E >, =, < 0$ as usual.

3. Lie and Noether Groups

Following the approach in Prince and Eliezer (1980), we consider infinitesimal point transformation

$$\bar{t} = t + \delta\alpha\, \xi(x, t), \qquad \bar{x}_i = x_i + \delta\alpha\, \eta_i(x, t) \tag{12}$$

generated by the operator

$$U \equiv \xi(\boldsymbol{x}, t)\, \partial/\partial t + \eta_i(\boldsymbol{x}, t)\, \partial/\partial x_i. \tag{13}$$

Induced variations in higher derivatives are displayed in the extended operators

$$U^{(n)} \equiv \xi \frac{\partial}{\partial t} + \eta_i \frac{\partial}{\partial x_i} + \eta'_i \frac{\partial}{\partial \dot{x}_i} + \ldots + \eta_i^{(n)} \frac{\partial}{\partial x_i^{(n)}} \tag{14}$$

where

$$\eta_i^{(k)}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ldots, \boldsymbol{x}^{(k)}, t) \equiv \mathrm{d}\eta_i^{(k-1)}/\mathrm{d}t - x_i^{(k)}\, \mathrm{d}\xi/\mathrm{d}t, \qquad k = 1, \ldots, n, \quad (15)$$

$\mathrm{d}/\mathrm{d}t$ being the total time derivative.

The finite transformations of the group can be determined by exponentiation of the infinitesimal operators

$$\bar{t} = \mathrm{e}^{\alpha U} t, \qquad \bar{x}_i = \mathrm{e}^{\alpha U} x_i, \quad (16)$$

where α is the group parameter, or by integration of the system of differential equations

$$\frac{\mathrm{d}\bar{t}}{\xi(\bar{x}, \bar{t})} = \frac{\mathrm{d}\bar{x}_1}{\eta_1(\bar{x}, \bar{t})} = \ldots = \frac{\mathrm{d}\bar{x}_n}{\eta_n(\bar{x}, \bar{t})} = \mathrm{d}\alpha \quad (17)$$

with initial conditions

$$\bar{t} = t, \qquad \bar{x}_i = x_i \qquad \text{when } \alpha = 0.$$

The Lagrangian being determined by the equation of motion only up to a total derivative, the invariance of the action function $\int \mathscr{L}\, \mathrm{d}t$ under (12) requires that ξ, η and an additional function $f(x, t)$ satisfy

$$U'\mathscr{L} + \dot{\xi}\mathscr{L} - \dot{f} = 0 \quad (18)$$

where U' is the first extension of U given by (14).

For each such ξ, η, f there is an associated constant of motion

$$J = (\xi\dot{x}_l - \eta_l)\, \partial\mathscr{L}/\partial\dot{x}_l - \xi\mathscr{L} + f. \quad (19)$$

The Lie method considers invariance of equations of motion of the form

$$\ddot{x}_i + g_i(\boldsymbol{x}, \boldsymbol{x}, t) = 0 \quad (20)$$

under point transformations (12). The condition for such invariance is

$$U''(\ddot{x}_i + g_i) = 0. \quad (21)$$

For the Kepler problem we will consider the motion to be in the xy plane so that

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + 0\boldsymbol{k}, \quad (22)$$

and

$$U \equiv \xi(x, y, t)\frac{\partial}{\partial t} + \eta(x, y, t)\frac{\partial}{\partial x} + \zeta(x, y, t)\frac{\partial}{\partial y}. \quad (23)$$

Using the Lagrangian (3) in (18) and equating coefficients of powers of $\dot{x}$ and $\dot{y}$ to zero, we obtain

$$-\frac{\mu}{r^3}(x\eta + y\zeta) + \frac{\mu}{r}\frac{\partial\xi}{\partial t} - \frac{\partial f}{\partial t} = 0, \quad (24)$$

$$\frac{\partial\eta}{\partial t} + \frac{\mu}{r}\frac{\partial\xi}{\partial x} - \frac{\partial f}{\partial x} = 0, \qquad \frac{\partial\zeta}{\partial t} + \frac{\mu}{r}\frac{\partial\xi}{\partial y} - \frac{\partial f}{\partial y} = 0, \quad (25)$$

$$2\frac{\partial\eta}{\partial x} - \frac{\partial\xi}{\partial t} = 0, \qquad 2\frac{\partial\zeta}{\partial y} - \frac{\partial\xi}{\partial t} = 0, \quad (26)$$

$$\frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial x} = 0, \qquad \frac{\partial\xi}{\partial x} = \frac{\partial\xi}{\partial y} = 0. \quad (27)$$

These are readily solved to yield

$$\xi = \text{constant}, \qquad \eta = -y, \qquad \zeta = x, \qquad f = \text{constant}, \tag{28}$$

which leads to the two independent generators

$$X_1 = \partial/\partial t, \qquad X_2 = x\,\partial/\partial y - y\,\partial/\partial x, \tag{29), (30}$$

of point transformations leaving the action integral invariant.

Using the equation of motion (2) with (22), we find that (21) is satisfied by the second extensions of both (23) and (24) and also by the second extension of the operator

$$X_3 = t\,\partial/\partial t + \tfrac{2}{3}x\,\partial/\partial x + \tfrac{2}{3}y\,\partial/\partial y. \tag{31}$$

This last operator does not satisfy the Noether requirement (18) that $U'\mathscr{L} + \dot{\xi}\mathscr{L}$ be a total time derivative.

The Lie algebra associated with these three operators is given by the commutators

$$[X_1, X_2] = 0, \tag{32}$$

$$[X_1, X_3] = X_1, \tag{33}$$

$$[X_2, X_3] = 0. \tag{34}$$

The Noether operators form a two-parameter abelian subgroup of the three-parameter Lie symmetry group.

Using the relations

$$[X_i, X_j] = C^k_{ij} X_k, \tag{35}$$

we see that the only non-zero structure constants C^k_{ij} are

$$C^1_{13} = 1 = -C^1_{31}. \tag{36}$$

The metric tensor of the algebra

$$g_{ij} = C^m_{ik} C^k_{jm} \tag{37}$$

is just

$$g_{ij} = \begin{cases} 1, & i = j = 3, \\ 0, & i, j \neq 3. \end{cases} \tag{38}$$

The algebra is non-semi-simple and is a semi-direct sum of the solvable subalgebra $\{X_1, X_3\}$ and the (trivially) simple subalgebra $\{X_2\}$.

4. Constants of motion

Using (9) for each of (29) and (30), Noether's theorem yields respectively

$$J_1 = \tfrac{1}{2}\dot{r}^2 + \mu/r, \tag{39}$$

$$J_2 = x\dot{y} - y\dot{x}, \tag{40}$$

the only non-zero component of angular momentum.

The method developed in Prince (1979) allows construction of constants associated with each element of the Lie group. In the case of the first two it produces (39) and (40) above. We outline the procedure for X_3.

Invariant functions of the extended group generated by X_3' are found by integrating

$$\frac{dt}{t}=\frac{dx}{\frac{2}{3}x}=\frac{dy}{\frac{2}{3}y}=\frac{d\dot{x}}{-\frac{1}{3}\dot{x}}=\frac{d\dot{y}}{-\frac{1}{3}\dot{y}}, \tag{41}$$

yielding

$$u_1(x,t)=x^3/t^2, \qquad u_2(y,t)=y^3/t^2 \tag{42}$$

and

$$v_1(x,\dot{x},t)=t\dot{x}^3, \qquad v_2(y,\dot{y},t)=t\dot{y}^3. \tag{43}$$

For the second extended group generated by X_3'' two further invariants are available, for example

$$\frac{dv_1}{du_1}\equiv\frac{\partial v_1/\partial t+\dot{x}\,\partial v_1/\partial x+\ddot{x}\,\partial v_1/\partial \dot{x}}{\partial u_1/\partial t+\dot{x}\,\partial u_1/\partial x} \tag{44}$$

and

$$\frac{dv_2}{du_2}\equiv\frac{\partial v_2/\partial t+\dot{y}\,\partial v_2/\partial y+\ddot{y}\,\partial v_2/\partial \dot{y}}{\partial u_2/\partial t+\dot{y}\,\partial u_2/\partial y}. \tag{45}$$

The most general pair of simultaneous second-order ordinary differential equations invariant under X_3'' is thus

$$dv_1/du_1=\phi_1(u_1,u_2,v_1,v_2), \tag{46}$$

$$dv_2/du_2=\phi_2(u_1,u_2,v_1,v_2), \tag{47}$$

where ϕ_1, ϕ_2 are arbitrary functions of their arguments.

Now we know that for motion in the xy plane the equations (2) must be expressible in the above form (using (42) and (45)) since (21) is satisfied. In fact (46) and (47) show how (2) may be reduced to a pair of first-order equations. These can then be integrated to give

$$W_1(u_1,u_2,v_1,v_2)=\text{constant}, \tag{48}$$

$$W_2(u_1,u_2,v_1,v_2)=\text{constant}. \tag{49}$$

Actual calculations are somewhat tedious and are displayed in the Appendix, the result being that (48) and (49) are

$$x\dot{y}^2-y\dot{x}\dot{y}-\mu x/r=\text{constant}, \tag{50}$$

$$y\dot{x}^2-x\dot{x}\dot{y}-\mu y/r=\text{constant}, \tag{51}$$

after (42) and (43) have been utilised. These are just the x and y components of the Runge–Lenz vector for motion in the xy plane.

Thus the X_3 operator of the Lie group yields the Runge–Lenz vector, while energy and angular momentum are obtained from operators in both Noether and Lie groups.

The whole problem may be treated in polar coordinates, whence the equation of motion (2) in component form is

$$\ddot{r}-r\dot{\theta}^2+\mu/r=0, \tag{52}$$

$$r\ddot{\theta}+2\dot{r}\dot{\theta}=0. \tag{53}$$

Noether group operators are

$$Y_1 = X_1 = \partial/\partial t, \tag{54}$$

$$Y_2 = \partial/\partial\theta. \tag{55}$$

The Lie approach provides the additional operator

$$Y_3 = t\,\partial/\partial t + r\,\partial/\partial r. \tag{56}$$

Constants of motion are again available via the methods used in the cartesian case. They are, for Y_1, Y_2, Y_3 respectively,

$$K_1 = J_1 = \tfrac{1}{2}\dot{r}^2 + \mu/r, \tag{57}$$

$$K_2 = J_2 = r^2\dot{\theta}, \tag{58}$$

$$\boldsymbol{K}_3 = \dot{r}\boldsymbol{e}_r + (r\dot{\theta} - \mu/r\dot{\theta})\boldsymbol{e}_\theta. \tag{59}$$

(59) is just Hamilton's vector (10).

5. Transformation of solutions

The finite transformations generated by (29), (30) and (31) can be calculated from (16) or (17) and are respectively

$$\bar{t} = t + \alpha_1, \qquad \bar{x} = x, \qquad \bar{y} = y, \tag{60}$$

$$\bar{t} = t, \qquad \bar{x} = x\cos\alpha_2 - y\sin\alpha_2, \qquad \bar{y} = x\sin\alpha_2 + y\cos\alpha_2, \tag{61}$$

$$\bar{t} = t\,\mathrm{e}^{\alpha_3}, \qquad \bar{x} = x\,\mathrm{e}^{2/3\alpha_3}, \qquad \bar{y} = y\,\mathrm{e}^{2/3\alpha_3}. \tag{62}$$

Now each of these sets of finite transformations takes a solution of (2) into another solution. The way this is accomplished in each case is illuminated by considering the transformation properties of the first integrals, keeping in mind the correspondence with the geometric characteristics of the solutions outlined in § 2.

We remark that if J_i is the constant of motion associated with an operator X_i then $X_i'J_i = 0$ (no sum). This is a direct consequence of the means by which the constant was obtained (Prince 1979). As X_i transforms solutions into solutions J_i is unchanged. This means that $\bar{J}_i = J_i$ under the finite transformation generating X_i.

For X_1,

$$\bar{E} = E, \qquad \bar{\boldsymbol{L}} = \boldsymbol{L}, \qquad \bar{\boldsymbol{R}} = \boldsymbol{R}. \tag{63}$$

Indeed, it is evident from (60) that a solution Γ of (2) is transformed into a geometrically identical solution Γ_1 on which the particle was started off a time α_1 later that the particle on Γ. The orbits differ only in their initial conditions.

For X_2,

$$\bar{E} = E, \qquad \bar{\boldsymbol{L}} = \boldsymbol{L}, \qquad \bar{R} = R, \qquad \bar{\theta} = \theta + \alpha_2, \tag{64}$$

where θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{R}$. The transformed orbit Γ_2 has the same intrinsic geometry as Γ, but the axis of the conic has been rotated through an angle α_2. Again the orbits differ only in their initial conditions.

For X_3,

$$\bar{E} = E\,\mathrm{e}^{-2/3\alpha_3}, \qquad \bar{h} = h\,\mathrm{e}^{1/3\alpha_3}, \qquad \bar{\hat{\boldsymbol{L}}} = \hat{\boldsymbol{L}}, \qquad \bar{\boldsymbol{R}} = \boldsymbol{R}. \tag{65}$$

The transformed orbit Γ_3 does not have the same intrinsic geometry as Γ. The eccentricity remains the same but the semi-latus rectum has changed. $\bar{l} = l\,\mathrm{e}^{2/3\alpha_3}$, using (8). The way in which the orbit is oriented in E_3 remains the same, however, so the transformations associated with the Runge–Lenz vector are not connected with initial conditions in the same way as those for the energy and angular momentum. The Runge–Lenz vector is a 'hidden symmetry' in this sense.

It is interesting to note that on elliptic orbits one finds that the period and semi-major axes transform as

$$\bar{P} = P\,\mathrm{e}^{\alpha_3}, \qquad \bar{a} = a\,\mathrm{e}^{2/3\alpha_3}, \tag{66}$$

and that P^2/a^3 is an invariant of the transformation (62). It is reassuring to know that the solutions related through (62) do not violate Kepler's third law.

6. The *n*-dimensional case

The results obtained in the previous sections are easily extended to the case of the equation of motion

$$\ddot{x}_i + \mu x_i/r^3 = 0, \tag{67}$$

where $r^2 = x_k x_k$. Energy and angular momentum are generalised to

$$E = \tfrac{1}{2}\dot{x}_k\dot{x}_k - \mu/r, \tag{68}$$

$$L_{ij} = x_i\dot{x}_j - x_j\dot{x}_i. \tag{69}$$

Ikeda and Maekawa (1970) generalised the Runge–Lenz vector in the following way. In three dimensions the cartesian components of $\boldsymbol{R}$ are given by

$$\begin{aligned} R_i &= \varepsilon_{ikl}\dot{x}_k L_l - \mu x_i/r \\ &= \varepsilon_{ikn}\varepsilon_{lmn}\dot{x}_k x_l \dot{x}_m - \mu x_i/r \qquad (i, j, \ldots = 1, 2, 3). \end{aligned} \tag{70}$$

Expanding this in terms of the Kronecker delta,

$$R_i = \tfrac{1}{2}(2\delta_{il}\delta_{km} - \delta_{im}\delta_{kl} - \delta_{ik}\delta_{ml})\dot{x}_k x_l \dot{x}_m - \mu x_i/r. \tag{71}$$

If we let the indices run from 1 to n, (71) may be considered the n-dimensional form of the Runge–Lenz vector.

Expressions (68), (69), (71) are all constants of the motion. The Lie method produces the following generators:

$$U_1 \equiv \partial/\partial t, \tag{72}$$

$$u_2^{ij} \equiv x_i\,\partial/\partial x_j - x_j\,\partial/\partial x_i, \tag{73}$$

$$U_3 \equiv t\,\partial/\partial t + \tfrac{2}{3}x_j\,\partial/\partial x_j. \tag{74}$$

It can be shown that (72), (73) and (74) produce (68), (69) and (71) respectively by the methods used earlier. It is somewhat easier to verify this:

$$U_1' E = 0, \qquad U_2^{ij\prime} L_{kl} = 0, \qquad U_3' R_i = 0. \tag{75}$$

The structure of the Lie algebra is straightforward:

$$[U_1, U_2] = U_1, \qquad [U_1, U_2^{ij}] = 0, \qquad [U_2^{ij}, U_3] = 0, \tag{76}$$

$$[U_2^{ij}, U_2^{kl}] = -\delta_{ik}U_2^{jl} + \delta_{il}U_2^{jk} + \delta_{jk}U_2^{il} - \delta_{jl}U_2^{ik}. \tag{77}$$

Some points regarding the angular momentum subgroup SO(n) were made by Prince and Eliezer (1980).

Appendix

Determination of the two components of the Runge–Lenze vector is facilitated by using

$$U_1 = xt^{-2/3}, \qquad U_2 = yt^{-2/3}, \tag{A1}$$

$$V_1 = \dot{x}t^{1/3}, \qquad V_2 = \dot{y}t^{1/3}, \tag{A2}$$

and in place of $\mathrm{d}v_1/\mathrm{d}u_1$, $\mathrm{d}v_2/\mathrm{d}u_2$ the three derivatives $\mathrm{d}U_2/\mathrm{d}U_1$, $\mathrm{d}V_1/\mathrm{d}U_1$, $\mathrm{d}V_2/\mathrm{d}U_2$. Direct calculation yields

$$\frac{\mathrm{d}U_2}{\mathrm{d}U_1} = \frac{3V_2 - 2U_2}{3V_1 - 2U_1} = \Phi(U_1, U_2, V_1, V_2), \tag{A3}$$

$$\frac{\mathrm{d}V_1}{\mathrm{d}U_1} = \frac{V_1 - 3\mu U_1 P^{-3}}{3V_1 - 2U_1} = \Psi(U_1, U_2, V_1, V_2), \tag{A4}$$

$$\frac{\mathrm{d}V_2}{\mathrm{d}U_1} = \frac{V_2 - 3\mu U_2 P^{-3}}{3V_1 - 2U_1} = \Theta(U_1, U_2, V_1, V_2), \tag{A5}$$

where

$$P = (U_1^2 + U_2^2)^{1/2} = rt^{-2/3}. \tag{A6}$$

Solutions of (A3), (A4) and (A5) are

$$W_1(U_1, U_2, V_1, V_2) = U_1V_2^2 - U_2V_1V_2 - \mu U_1/P = \text{constant}, \tag{A7}$$

$$W_2(U_1, U_2, V_1, V_2) = U_2V_1^2 - U_1V_1V_2 - \mu U_2/P = \text{constant}, \tag{A8}$$

as may be seen by verifying that

$$\frac{\mathrm{d}W_1}{\mathrm{d}U_1} = \frac{\partial W_1}{\partial U_1} + \Phi\frac{\partial W_1}{\partial U_2} + \Psi\frac{\partial W_1}{\partial V_1} + \Theta\frac{\partial W_1}{\partial V_2} = 0 \tag{A9}$$

and similarly for W_2. Use of (A1) and (A2) in (A7) and (A8) yields (51) and (52).

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